Smith-Pursell radiation from different kinds of gratings. Comparison of theoretical models and experimental results.

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Investigation problems

- CSPR from different profile targets – comparison and simulations based on the Van den Berg model, surface current model and resonant diffraction radiation one.
- Pre-wave zone effect for SPR
- Experimental results on coherent SPR
- Estimation of the electron bunch length
Incoherent Smith-Purcell radiation (SPR):

First observed by S.J. Smith and E.M. Purcell, Phys. Rev. Lett. 92, 1069 (1953)

Smith-Purcell dispersion relation:

\[ \lambda = \frac{d}{n} \left( 1 - \cos[\Theta] \right) \]
Possibilities of SPR application:


2. Compact free electron lasers based on SPR in millimeter and sub-millimeter range (V. Kumar and K.-J. Kim, Phys. Rev. E 73, 026501 (2006)).

There are some fuzzy points of SPR features need to clarify:

1. There are many different models exist to describe features of SPR from gratings of different profile.

   **What model is correct for \( E_e \leq 10 \text{ MeV} \) and chosen grating profile?**

   **What kind of grating profile may provide the best “coupling” between beam and grating?**

2. All the SPR models describe SPR features in so-called “wave” (or “far”) zone where the radiation source can be considered as point-like.

   **When does this approximation correct?**

   **What are the main differences of SPR properties in “wave” and “pre-wave” zones (where this approximation is not valid)?**
Let us compare SPR models. We will consider the most often used of them:

1. **Van den Berg’s model** (P.M. van den Berg, J. Opt. Soc. Am. 63, No.12, 1588-1597 (1973)) – the relativistic particle field is substituted for a packet of evanescent waves which are diffracting on the grating according to optics laws.


*Difference between SPR intensity calculated using models №1 and №2, №3 ones may achieve 2 orders of magnitude for some gratings and $E_e \leq 10$ MeV*
**Experiment investigations of SPR**

The experiments those authors came to conclusion about accordance of their data with theoretical predictions of Surface Current model:


The experiments those authors came to conclusion about accordance of their data with theoretical predictions of van den Berg’s model:

Grating profiles we will consider:

- **“Volume” gratings**
- **Flat strip grating**
Angular distribution of SPR per an electron per one grating period:

Van den Berg’s model:

\[
\frac{dW_n}{d\Omega} = \frac{\alpha \hbar c}{2d} n^2 \frac{\sin^2 \Theta \cos^2 \Phi}{\left(1/\beta - \cos \Theta\right)^3} |\mathcal{R}_n|^2 \times \exp\left[-\frac{h}{h_{\text{eff}}} \sqrt{1 + (\beta \gamma \sin \Theta \sin \Phi)^2}\right].
\]

Surface current model:

\[
\frac{dW_n}{d\Omega} = \frac{\alpha \hbar c}{d} \frac{2\pi n^2}{\left(1/\beta - \cos \Theta\right)^3} |\vec{R}_n|^2 \times \exp\left[-\frac{h_2}{h_{\text{eff}}} \sqrt{1 + (\beta \gamma \sin \Theta \sin \Phi)^2}\right],
\]

\[
|\vec{R}_n|^2 = \left|\vec{n} \times [\vec{n} \times \vec{G}]\right|^2.
\]

RDR model:

\[
\frac{d^2 W_{RDR}}{d\omega d\Omega} = \frac{d^2 W_{DR}}{d\omega d\Omega} F_{n,\text{cell}} F_N
\]
Comparison of surface current (SC) and RDR model

\[
\frac{dW_1}{d\Omega} \times 10^{-8} \frac{eV}{sr}
\]

\(\gamma = 12, d = 8\text{mm}, a = 4\text{mm}, h = 10\text{mm}, \Theta_\theta = 0^\circ, \Phi = 0^\circ\)

\(a.)\)

\[
\frac{dW_1}{d\Omega} \times 10^{-8} \frac{eV}{sr}
\]

\(\gamma = 12, d = 8\text{mm}, a = 4\text{mm}, h = 10\text{mm}, \Theta_\theta = 90^\circ, \Phi = 0^\circ\)

\(b.)\)

\[
\frac{dW_1}{d\Omega} \times 10^{-8} \frac{eV}{sr}
\]

\(\gamma = 12, d = 8\text{mm}, a = 4\text{mm}, h = 10\text{mm}, \Theta_\theta = 26^\circ, \Phi = 20^\circ\)

\(a.)\)

\[
\frac{dW_1}{d\Omega} \times 10^{-8} \frac{eV}{sr}
\]

\(\gamma = 12, d = 8\text{mm}, h = 10\text{mm}, \Theta_\theta = 26^\circ, \Theta = 90^\circ\)

\(b.)\)
Comparison of van den Berg’s model and surface current model (RDR)

\[ \frac{dW_1}{d\Omega} \left( \frac{eV}{sr} \right) \]

\[ \gamma = 12, \]
\[ d = 8\,mm, \]
\[ a = d/2, \]
\[ h = 10\,mm, \]
\[ \Phi = 0 \]

**FIG. 6:** The angular distribution of the SPR intensity for a "thin" grating with vacuum gaps: a solid line for the RDR model; a dot line \((b/d = 1/8)\) and a dash-dotted line \((b/d = 0.001)\) for van den Berg’s model.
Comparison of van den Berg’s model and surface current model (RDR)

\[
\frac{dW_1}{d\Omega} \left( \frac{eV}{sr} \right)
\]

\[
\begin{align*}
\gamma &= 12, \\
d &= 8\text{mm}, \\
a &= d/2, \\
h &= 10\text{mm}, \\
\Phi &= 0
\end{align*}
\]

FIG. 7: Angular distribution of the SPR intensity for the lamellar grating: according to the current model - a solid line \((b/d = 1/8)\), according to van den Berg’s model - a dot line \((b/d = 1/8)\) and a dash-dotted line \((b/d = 0.001)\).
Comparison of van den Berg’s model and surface current model (RDR)

\[
\frac{dW}{d\Omega} \times 10^{-11} \left( \frac{eV}{sr} \right)
\]

\[\gamma = 12, \quad d = 8 \text{ mm}, \quad a = d/2, \quad \frac{b}{d} = 0.001, \quad h = 10 \text{ mm}, \quad \Theta = 90^\circ\]

FIG. 11: Azimuth dependence of the SPR intensity for a "flat" grating according to van den Berg’s model (a solid line) and according to the RDR model (a dot line, should be multiplied by \(10^3\)).
\[ \frac{dW}{d\Omega} \times 10^{-7}, \frac{eV}{sr} \]

**a.)**

\[ \frac{dW}{d\Omega} \times 10^{-12}, \frac{eV}{sr} \]

\[ b/d = 0.001 \]

\[ h = h_1 \]

\[ \gamma \]

**b.)**

\[ d = 8mm, \]
\[ a = 4mm, \]
\[ h = 2mm, \]
\[ \Theta = 120^\circ, \]
\[ \Phi = 0, \]
\[ a/d = 0.001 \]
Conclusions from SPR models comparison:

1. There exists a difference in the SPR intensity between the "volume" and "flat" gratings: the SPR yield from the "volume" grating predicted by van den Berg's model for a moderately relativistic case is by two orders of magnitude higher practically for all the polar angles;

2. There exists a difference in the azimuth dependencies for the "flat" gratings: the current and RDR models predict a single-maximum angular distribution with maximum in the plane perpendicular to the grating (Φ = 0), while van den Berg's model predicts the distribution with a minimum in the plane Φ = 0;

3. There exists a difference in the SPR yield dependence on the Lorentz-factor of a particle: van den Berg's model predicts the yield decrease with the energy growth, while the current and RDR models predict the yield increase.
2. Smith-Purcell radiation in “pre-wave” zone

The unusually large distance corresponds to “far” zone approximation for X-FELs based on undulator radiation. That is because of theirs very long insertion devices. For example, the SLAC insertion device to Linac Coherent Light Source (LCLS) has 140 m length and minimal distance corresponds to “far” zone is about 380 m (R. Tatchyn, Proc. 27th Int. Free Electron Laser Conf., 21-26 August 2005, Stanford, USA, P.282).

The term “pre-wave zone” was considered for the first time by V.A. Verzilov, Phys. Letters A 273, 135-140 (2000). It was shown that for the case of backward transition and diffraction radiation (BTR and BDR respectively) the “far”-zone criterion \( R_0 \geq \gamma^2 \lambda \) \( R_0 \)- distance between target and detector centers
SPR geometry
“Far”-zone criterion for SPR

The longitudinal size of radiating surface is equal to grating length \( L \) (a). The transversal size is equal to grating width \( M \).

Let us consider two waves radiated from opposite sides of grating (a) when a charged particle goes near.

With approximation of far distance from detector (that is much more than grating length) one can write phase difference in the following form:

\[
\Delta \varphi = \varphi_B - \varphi_A \approx k \left( r_2 - r_1 + \frac{L}{\beta} \right),
\]
“Far”-zone criterion for SPR

Using following expression

\[ \vec{r}_2 = \vec{r}_1 - \vec{L}, \]

one can obtain:

\[ \Delta \varphi \approx k \left( \frac{L^2}{2r_1} - \frac{(\vec{L}, \vec{r}_1)}{r_1} + \frac{L}{\beta} \right), \]

The two last terms do not depend on the detector distance with respect to the grating unlike the first term that is the first-order correction of “pre”-wave zone.

So, the “far”-zone criterion will be:

\[ k \frac{L^2}{2r_1} \ll \pi \]

or:

\[ r_1 \gg \frac{L^2}{\lambda_n} \approx N^2 d \frac{n}{\beta^{-1} - \cos \Theta} \]
“Far”-zone criterion for SPR

The same criterion may be obtained for transversal SPR distributions (in case when the grating width less than particle coulomb field radius):

\[ r'_1 \gg \frac{M^2}{\lambda_n} \approx M^2 \frac{n}{d(\beta^{-1} - \cos \Theta)} \]

Thus the «far»-zone condition for SPR in relativistic case does not depend on particle energy!
Let us consider the influence of detector disposition in “pre-wave” zone on the SPR angular distributions. For that we will use the model proposed by V.A. Verzilov in already cited work. In this model the relativistic particle field is expressed by packet of plane waves. The field components on the detector plane will be written in the form:

\[
\begin{pmatrix}
E^D_x \\
E^D_z
\end{pmatrix}
= const \int_{-M/2}^{M/2} dX_T \int_{-Nd/2}^{Nd/2} dZ_T \left( \frac{X_T}{\hbar} \right) \chi(Z_T) \times K_1 \left[ \frac{2\pi}{\beta\gamma\lambda} \sqrt{\frac{X_T^2 + h^2}{X_T^2 + h^2}} \right] \exp \left[ i\Delta\varphi(X_T, Z_T, X_D, Z_D) \right].
\]

Radiation intensity will find as usual:

\[
I = const(\left| E^D_x \right|^2 + \left| E^D_z \right|^2)
\]
Let us obtain the expression for phase shift:

$$\Delta \phi = \frac{2\pi}{\lambda} \left( \Delta \mathcal{R} + \frac{Z_T}{\beta} \right)$$

$$\Delta \mathcal{R} = \mathcal{R}(Z_T, X_T, X_D, Z_D) - \mathcal{R}(0, 0, X_D, Z_D)$$

$$\Delta \phi = \frac{2\pi}{\lambda} \left( \left[ \mathcal{R}_0^2 + (X_T - X_D)^2 + Z_T^2 + Z_D^2 + \right.$$  
$$\left. - 2Z_T(\mathcal{R}_0 \cos \Theta + Z_D \sin \Theta) \right]^{1/2} -$$  
$$\left. - \left[ \mathcal{R}_0^2 + Z_D^2 + X_D^2 \right]^{1/2} + \frac{Z_T}{\beta} \right),$$
Derivation of expression for the phase shift

Assume that:
\[ \mathcal{R}_0 \gg X_T, Y_T, X_D, Y_D \]

And using transformation of variables:
\[
\begin{pmatrix}
  x_T \\
  z_T \\
  l
\end{pmatrix}
= \frac{2\pi}{\beta \gamma \lambda}
\begin{pmatrix}
  X_T \\
  Z_T \\
  L
\end{pmatrix},
\]
\[
\begin{pmatrix}
  x_D \\
  z_D \\
  l
\end{pmatrix}
= \frac{\beta \gamma}{\mathcal{R}_0}
\begin{pmatrix}
  X_D \\
  Z_D \\
  L
\end{pmatrix},
\]
\[
R = \frac{\mathcal{R}_0}{L^2/\lambda} = 4\pi^2 \frac{1}{l^2} \frac{\mathcal{R}_0}{\gamma^2 \lambda}.
\]

We obtain more simple expression for the phase shift:
\[
\Delta \varphi \approx \pi \frac{z_T^2 + x_T^2}{l^2 R} - \beta \gamma z_T \cos \Theta - z_T z_D \sin \Theta - x_T x_D + \gamma z_T.
\]
The results obtained

D.V. Karlovets, A.P. Potylitsyn PRST-AB (2006), to be published
SPR focusing

\[ r = 100 \cdot R_0 \quad (\text{almost flat grating}), \]
\[ r = 10 \cdot R_0, \]
\[ r = 2 \cdot R_0 \]

\[ \gamma = 10, \quad d = 5 \text{ mm}, N = 9, \]
\[ \Theta = 90^\circ \quad (\lambda \approx 5 \text{ mm}), \]
\[ h = 10 \text{ mm}, \]
\[ M (\text{flat grating width}) = 20 \text{ mm}, \]
\[ R_0 = 20 \text{ mm}, \]
\[ R \approx \left( \frac{R_0 \lambda}{M} \right) \approx 0.25 \]

**Focusing effect for SPR from concave cylindrical grating**
The results obtained

The approach developed allows to calculate SPR characteristics from “concave” grating.

One may expect that for a “cylindrical” strip grating there may exist the “focusing” effect.

The experimental verification of the model proposed allows to choose kind of grating (N, d, etc.) beam energy and impact parameter in order to receive the maximal SPR power at the fixed detector position.

So, from figures one can see that with decrease of distance to detector
**Experimental scheme**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron energy</td>
<td>6.1 MeV</td>
</tr>
<tr>
<td>Macropulse duration</td>
<td>2 – 6 µs</td>
</tr>
<tr>
<td>Pulse repetition rate</td>
<td>1 – 8 Hz</td>
</tr>
<tr>
<td>Bunch length $\sigma$ (Gauss approximation)</td>
<td>~1.3 mm</td>
</tr>
<tr>
<td>Number of electrons per bunch</td>
<td>~$10^8$</td>
</tr>
<tr>
<td>Number of bunches per macropulse</td>
<td>~$10^4$</td>
</tr>
<tr>
<td>Beam size at the microtron output</td>
<td>4×2 mm²</td>
</tr>
<tr>
<td>Emittance:</td>
<td></td>
</tr>
<tr>
<td>horizontal</td>
<td>3·$10^{-2}$ mm×rad</td>
</tr>
<tr>
<td>vertical</td>
<td>1.5·$10^{-2}$ mm×rad</td>
</tr>
</tbody>
</table>
Coherent Smith-Purcell radiation (CSPR):

Spectral-angular density of radiation:

\[
\frac{dW_{\text{CSPR}}(\omega)}{\hbar d\omega d\Omega} = N_e \left[1 + N_e \cdot f(\omega)\right] \frac{dW_1(\lambda)}{\hbar d\omega d\Omega}
\]

- \(N_e\) - number of electrons per bunch,
- \(dW_1(\lambda)\) - spectral-angular density of SPR from one electron,
- \(f(\omega)\) - “so-called” form-factor depending on radiation frequency, bunch shape and particles distribution functions in bunch.

One can find that for the case of \(\lambda > \sigma_l\) (\(\sigma_l\) — bunch length), intensity of SPR increases \(\sim N_e\) by times.
Based on the broadband antenna with the high frequency diode.

**Detector operates at a room temperature.**

*Wavelength region: $\lambda = \text{3~20 mm}$  
Sensitivity: $\approx 0.3\text{V/mWatt}$*

*Wave-guide d=10 mm,  
passes wavelengths $\lambda < 17 \text{ mm}$*  
[K. Hanke, DESY, CLIC Note 298, 19.04.1996]
Targets

Aluminium

Copper

Copper on plastic
Scheme of experiment
Azimuthal CSPR distribution

Flat SPR target on the dielectric Dependence on impact-parameter

Radiation Yield / Q
Faraday cup reading Q normalized by beam current
CSPR azimuthal distribution from different target profiles

Azimuthal distribution for different impact-parameters
Triangular strip target

- $\phi$ (degree)
- Yield/Q

- $h = 7\text{ mm}$
- $h = 27\text{ mm}$
Azimuthal CSPR distribution (theory)

Azimuthal dependences were calculated for 3 kinds of gratings:


2. Volume strip grating (see, for instance, G.Kube. NIM B 227 (2005),180-190),


Up to now there is no models for calculations of SPR characteristics from flat grating with dielectric gaps.
Azimuthal CSPR distribution (theory)

The intensities ratio of SPR from these gratings

\[ \Theta = 30^\circ, \Phi = 90^\circ : \]

\[ \frac{dW_{1_{\text{flat}}}}{d(h\omega)d\Phi} : \frac{dW_{1_{\text{Vol.(Above)}}}}{d(h\omega)d\Phi} : \frac{dW_{1_{\text{Lam.}}}}{d(h\omega)d\Phi} \approx 1 : 0.093 : 0.0068 \]

Thus, the most effective is the flat grating

Setup for angular distributions measurements

Aperture of the telescope

\[ \Delta \theta = \frac{\phi_d}{2f} = \frac{10}{2 \cdot 151} = 33 \text{ mrad} \]
Flat SPR target. Cupper on the dielectric.
Polar dependence (Beam in). h=7mm.

Coherence threshold

Flat SPR target. Cupper on the dielectric.
Polar dependence (Beam out). h=-12mm.
Flat aluminum SPR target with air gaps. $h=9\text{mm}$.

<table>
<thead>
<tr>
<th>$\theta$ (degree)</th>
<th>Rad. Yield / Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>3.0</td>
</tr>
<tr>
<td>40</td>
<td>2.5</td>
</tr>
<tr>
<td>60</td>
<td>2.0</td>
</tr>
<tr>
<td>80</td>
<td>1.5</td>
</tr>
<tr>
<td>100</td>
<td>1.0</td>
</tr>
<tr>
<td>120</td>
<td>0.5</td>
</tr>
</tbody>
</table>

SPR target with triangular strips. $h=7\text{mm}$.

<table>
<thead>
<tr>
<th>$\theta$ (degree)</th>
<th>Rad. Yield / Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.6</td>
</tr>
<tr>
<td>40</td>
<td>1.4</td>
</tr>
<tr>
<td>60</td>
<td>1.2</td>
</tr>
<tr>
<td>80</td>
<td>1.0</td>
</tr>
<tr>
<td>100</td>
<td>0.8</td>
</tr>
<tr>
<td>120</td>
<td>0.6</td>
</tr>
</tbody>
</table>
CSPR angular distribution

d=12 mm

Theoretical estimations

\[ P_{vdB} \approx 0.09 \frac{mWatt}{sr} \]

\[ P_{RDR} \approx 3000 \frac{mWatt}{sr} \]
CSPR spectrum

d=12mm, N=14, h=8 mm, \( \theta = 100^\circ \)

\( 0^\circ \leq \alpha \leq 90^\circ \)
CSPR spectrum
Setup for CTR angular distributions measurements

Aperture of the telescope

\[ \Delta \theta = \frac{\phi_d}{2f} = \frac{10}{2 \times 1.51} = 33 \text{ mrad} \]
Coherence criterion

\[ Y = A \cdot X^{1.93 \pm 0.03} \]

CTR & CSPR comparison
Maximal yield from different targets at $\varphi=0$

<table>
<thead>
<tr>
<th>Target</th>
<th>Max. yield (arb.un.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TR target</td>
<td>0.87</td>
</tr>
<tr>
<td></td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>0.07</td>
</tr>
<tr>
<td><strong>TR target</strong></td>
<td><strong>3.8</strong></td>
</tr>
</tbody>
</table>

Bunch length estimation

Form-factor

\[ |f(\lambda)|^2 = \text{Exp}\left\{-\frac{4\pi^2\sigma^2}{\beta^2\lambda^2}\right\} \]

\[ L_{\text{bunch}} = 6\sigma \]

\[ \sigma \approx 1.13 \text{ mm} \]
Conclusion

1. For moderately relativistic electron beam the flat target is most effective for CSPR generation (see upper table).
2. For fixed impact parameters the azimuthal distribution of CSPR from a flat target has a maximum in the plane perpendicular grating.
3. From angular distribution of CSPR it is possible to determine the bunch length using a broadband detector.
4. For small polar angles ($\theta<40^\circ$) we observed large contribution of coherent diffraction radiation from entrance and exit edges of target as whole.
5. Resonant diffraction radiation model theoretical estimations are in the better agreement with experimental results than van den Berg’s model ones.