Wakefields and beam hosing instability in plasma wake acceleration (PWFA)

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Introduction to PWFA

- Plasma wake excited by relativistic particle bunch
- “Blow-out” regime when $n_b/n_e > 1$
- Acceleration and focusing by plasma
- Accelerating field scales as $n_e^{1/2}$
- Typical: $n_e \sim 10^{17}$ cm$^{-3}$, $k_p^{-1} = 17$ µm, $E \gtrsim 10$ GV/m, $G \gtrsim$ MT/m

C. Joshi, W.B. Mori, (2006)
Future applications

10 TeV PWFA-LC concept (J.P. Delahaye, et al. IPAC-2014).


Also many others: FlashForward, Eupraxia, ...
Are there principal limitations for plasma-based LC?

Comment on “Beamstrahlung considerations in laser-plasma-accelerator-based linear colliders”

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Schroeder, Esarey, Geddes, Benedetti, and Leemans [Phys. Rev. ST Accel. Beams 13, 101301 (2010) and Phys. Rev. ST Accel. Beams 15, 051301 (2012)] have proposed a set of parameters for a TeV-scale collider based on plasma wakefield accelerator principles. In particular, it is suggested that the luminosities greater than $10^{34} \, \text{cm}^{-2} \, \text{s}^{-1}$ are attainable for an electron-positron collider. In this Comment we dispute this set of parameters on the basis of first principles. The interactions of accelerating beam with plasma impose fundamental limitations on beam properties and, thus, on attainable luminosity values.

The claim: “In summary, we believe that the collider parameters, presented in Refs. [1,2], are not self-consistent. We would also like to note that our attempts to correct the above problems by adjusting the parameters while keeping the same overall performance (i.e., beamstrahlung, luminosity, and power consumption) were unsuccessful.”

Hosing instability in PWFA

Courtesy of Weiming An from UCLA. Parameters of the simulations: the driver has $\sigma_z = 12.77$ µm, $\sigma_r = 3.65$ µm, $Q = 1.6$ nC, ($I_{\text{peak}} = 15$ kA); the witness has $\sigma_z = 6.38$ µm, $\sigma_r = 3.65$ µm, $Q = 0.69$ nC, ($I_{\text{peak}} = 13$ kA). Plasma density $4 \times 10^{16}$ cm$^{-3}$. The distance between the bunches is a) 108 µm and b) 150 µm.

To study the hosing (beam-breakup) instability in the blowout regime we need to know wakefields inside the bubble.
Plasma wakefields

The terminology of wakefields in plasma can be confusing. The original meaning of the wake in plasma is the field generated by the *driver* that accelerates the *witness* beam. The driver is a beam of charged particles (PWFA) or a laser beam (LWFA).

In this presentation, by wakefields I mean the fields (longitudinal and transverse) with which the *witness bunch* acts on itself. They are generated by the *leading* charges and act on the *trailing* charges of the witness bunch.

In *linear approximation*, valid for $n_b \ll n_p$, one can assume that the perturbation of the plasma density is small, $\delta n_e \ll n_e$. The wakefield problem can be solved analytically for arbitrary charge distribution of the driver and witness bunches\(^1\). This approach, unfortunately, does not work in the blowout regime.

\(^1\) T. Katsouleas et al., Particle Accelerators, 22, 81 (1987).
Plasma equations

This is the system of equations (in dimensionless units) that governs the plasma dynamics in \textit{axisymmetric} geometry. We assume beams moving with \( v = c \), a steady state with everything depending on \( \xi = ct - z \) and \( r = \sqrt{x^2 + y^2} \).

Introduce \( \psi = \phi - A_z \),

\[
E_z = \partial_\xi \psi, \quad E_r = -\partial_r \psi
\]

Eq. for \( \psi \)

\[
\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \psi = n_e (1 - v_z) - 1
\]

Eq. for \( B_\theta \)

\[
\frac{1}{r} \frac{\partial}{\partial r} r B_\theta = -\frac{\partial}{\partial \xi} n_e v_r - \frac{\partial}{\partial r} n_e v_z - \frac{\partial n_d}{\partial r} - \frac{\partial n_w}{\partial r}
\]

Eqs. of motion for plasma electrons

\[
\frac{dp_r}{d\xi} = \frac{\gamma}{1 + \psi} \partial_r \psi - B_\theta, \quad \frac{dr}{d\xi} = \frac{p_r}{1 + \psi}, \quad 1 - v_z = \frac{1}{\gamma} (1 + \psi)
\]

The continuity equation

\[
\partial_\xi [n_e (1 - v_z)] + \frac{\partial}{\partial r} r n_e v_r = 0
\]

\[
\omega_p = \sqrt{4\pi n e^2 / m}
\]

\[
\xi \rightarrow \xi k_p \frac{-1}{\alpha}
\]

\[
E \rightarrow E m c \omega_p / e
\]
Relativistic point charge moving in free space

In wakefield theory for relativistic beams we assume $v = c$. When a point charge $q$ is moving in vacuum, its field is

$$E_r = B_0 = \frac{2q}{r} \delta(z - ct)$$

What happens if the point charge is moving in uniform, cold plasma of density $n_0$?
Point charge moving through plasma

The remarkable result of Ref.\textsuperscript{2} is the existence of the \textit{electromagnetic shock wave} (EMSW)

\[ E_r = B_\theta = 2qk_pK_1(k_pr)\delta(z - ct) \]

where \( k_p = \omega_p/c = \sqrt{4\pi n_0 e^2/mc^2} \) and \( K_1 \) is the modified Bessel function. For \( r \ll k_p^{-1} \) we recover \( E_r, B_\theta \approx 2q\delta(z - ct)/r; \) for \( r \gg k_p^{-1} \) the field decays exponentially, \( E_r, B_\theta \propto e^{-k_pr}/\sqrt{k_pr} \). Remarkably, the fields in EMSW are linear functions of the charge.

The only external dimensionless parameter in the problem is

\[ \nu = \frac{q}{e}r_e k_p = N_d r_e k_p \sim q\sqrt{n_0} \]

For \( n_0 = 10^{16} \ \text{cm}^{-3}, \ q = 1 \ \text{nC} \) we have \( k_p^{-1} = 53 \ \mu \text{m}, \ \nu = 0.3. \)

\textsuperscript{2} N. Barov et al., PRAB 7, 061301 (2004).
Plasma behind the relativistic point charge, $\nu \ll 1$

We normalize time to $\omega_p^{-1}$, length to $k_p^{-1}$ and fields to $mc\omega_p/e$, with $\xi = k_p( ct - z )$.

Plasma flow behind a point charge with $\nu \ll 1$ has been analyzed in Ref.\textsuperscript{3}. Immediately behind the charge one can use \textit{ballistic approximation} for the electron orbits. The \textit{bubble boundary}, $r_b(\xi)$, is found as an envelope for the family of trajectories:

$$r_b(\xi) = 2\sqrt{2\nu \xi}$$

Outside of the bubble there are two electron streams at each point $r, \xi$, corresponding to two different trajectories passing through this point. The plasma density has a singularity $n \propto (r - r_b)^{-1/2}$ as $r \to r_b$.

\textsuperscript{3} G. Stupakov et al. PRAB, \textbf{19}, 101302 (2016).
Trajectories and the bubble, $\nu \ll 1$

The bubble radius $r_{bm} = 2.82\sqrt{\nu}$ (the maximal value or $r$ on the boundary) is at $\xi_m = 1.57$. The total bubble length is $\xi_b = 3.8$. There are exactly two electron trajectories passing though each point $(r, \xi)$ outside of the bubble. The bubble boundary before its maximum $r_{bm}$ ($\xi < \xi_m$) is comprised of many trajectories for which the boundary is an envelope.

In Ref. 4 a differential equation for the bubble boundary was derived assuming that the boundary coincides with an electron trajectory. That model is satisfactory for drivers of finite size ($\sigma_z k_p \sim 1$) and large charges ($\nu \gtrsim 1$). It is not applicable to our case.

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Numerical solution of the bubble for arbitrary $\nu$

We developed a matlab code\textsuperscript{5} that solves an axisymmetric plasma bubble generated by a Gaussian driver and witness bunches. The code uses a novel fast algorithm of solving the nonlinear plasma flow. Illustrations: the driver with $Q = 1$ nC, $\sigma_z = 13 \, \mu$m, $\sigma_r = 5 \, \mu$m, plasma density $4 \times 10^{16} \, \text{cm}^{-3}$ ($k_p^{-1} = 26 \, \mu$m).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plots.png}
\caption{Plots of the longitudinal electric field. One unit of electric field is 19.2 GV/m.}
\end{figure}

Here \( F_z \) and \( F_r \) are integrated forces with which the leading charge \( (q_1) \) acts on the trailing one \( (q) \).

For relativistic beams \( (\gamma \gg 1) \) we usually assume that \( v = c \).
Wakefields in accelerator physics

Due to the linearity of Maxwell’s equations and the linearity of the boundary conditions, wakefields are linear functions of the charge $q_1$. We can calculate the wakefield of a bunch of particles by adding the wakes of its charges.

It was observed that the wake inside a round pipe of radius $a$ at a short distance behind the leading charge has a simple universal form (this is valid for resistive wall wake, a dielectric pipe, a periodic RF structure with $a$ the minimal aperture)

$$w_l(z) = \frac{4}{a^2} h(z) \quad w_t(z) = \frac{8z}{a^4} h(z)$$

where $h(z)$ is the step function (in SI system of units multiply by $Z_0c/4\pi$).

For a given wake we can start solving the beam stability problems.
In the absence of theory some researchers\textsuperscript{6} used for short-range \textit{wakefields} formulas in which they replaced the pipe radius $a$ by the bubble radius $r_b$ at the location of the source charge

\[ w_\ell(z) = \frac{4}{r_b^2} h(z) \quad w_t(z) = \frac{8z}{r_b^4} h(z) \]

We developed wake theory based on an accurate solution of plasma equations immediately behind the source charge. In this theory both the longitudinal and transverse wakes can be calculated\textsuperscript{7}.  

There is a jump in $E_z$ immediately behind the witness charge, $\Delta E_z(r, \xi)$. Remarkably, the theory predicts that this jump is proportional to the (dimensionless) witness charge $\nu_w$ (the charge has not to be small). So one can introduce the longitudinal wake is $w_\ell = \Delta E_z(0, \xi)/\nu_w$. 

Longitudinal wake in the bubble
Calculation of the longitudinal wake

First, one needs to calculate the strength of the EMSW, $D(r, \xi)$, at the location of the witness charge:

$$E_r(r, \xi) = D(r, \xi_0) \delta(\xi - \xi_0)$$

(here $\xi_0$ is the position of the source charge in the bubble). It satisfies the following equations

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} rD = \frac{n_{e0}(r, \xi_0)}{\gamma_0(r, \xi_0)} D$$

Here $n_{e0}$ and $\gamma_0$ are the quantities in the bubble without the witness charge. Then

$$\Delta E_z = -\frac{1}{r} \frac{\partial}{\partial r} rD$$

This result can be benchmarked against the wakefields in a hollow plasma channel.
Wakefields in a hollow plasma channel

Wakefields for a hollow plasma channel were calculated in\(^8\) in linear approximation (small charge limit).

Longitudinal and transverse wakes at short distance

\[
\begin{align*}
    w_\ell(+0) &= \frac{4}{a^2} \frac{K_0(ak_p)}{K_2(ak_p)} \\
    w_t'(+0) &= \frac{8}{a^4} \frac{K_1(ak_p)}{K_3(ak_p)}
\end{align*}
\]

This wake is reproduced by solving equations from the previous slide with \(n_{e0}(r) = n_0 h(r - a)\) and \(\gamma_0(r) = 1\). The wakes \(w_\ell(+0)\) and \(w_t'(+0)\) are valid not only in the linear, but in nonlinear regime as well.

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Longitudinal wake as a function of $\xi$,

This wake is in good agreement with the simulated jump in $\Delta E_z$ of a witness charge on the axis of the bubble. Red dashed line is a fit

$$w_\ell = \frac{4}{r_b^2}$$

$$w_\ell = \frac{4}{(r_b(\xi) + 0.8k_p^{-1})^2}$$
The source charge is now off axis, the offset is assumed small. The shock wave is not axisymmetric, \( E_r \propto \hat{D}(r, \xi) \cos \theta \), \( E_\theta \propto \hat{D}(r, \xi) \sin \theta \). Behind the wave \( \Delta E_z(r, \xi, \theta) = \Delta \hat{E}_z(r, \xi) \cos \theta \). The fields satisfy the following equations

\[
\partial_{rr}\hat{D} + \frac{1}{r} \partial_r \hat{D} - \frac{4\hat{D}}{r^2} = \frac{n_{e0}(r, \xi_0)}{\gamma_0(r, \xi_0)} \hat{D}
\]

\[
\Delta \hat{E}_z = -\partial_r \hat{D} - \frac{2\hat{D}}{r}
\]

The transverse wake is \( w_t \) is found from the Panofsky-Wenzel relation and it is a linear function of the distance between the source and the witness, \( w_t = w'_t(\xi_1 - \xi) \). Our result agrees with the linear approximation of the transverse wake in a plasma channel calculated by Schroeder et al.
Transverse wake as a function of $\xi$,

Red dashed line is a fit

$$w'_t = \frac{8}{r_b^4}$$

$$w'_t = \frac{8}{[r_b(\xi) + 0.75k_p^{-1}]^4}$$
The fit works well for different charges of driver bunches. The red dot-dashed lines are fitted curves.

Longitudinal wake (left panel) and the slope of the transverse wake (right panel) for 2 nC (label 1) and 4 nC (label 2) driver bunches.
BBU instability of the witness bunch

Having found wakefields in the plasma bubble, we can use them to study the beam-breakup instability of the witness bunch\(^9\).

\[
\begin{align*}
X(s, z) &= \frac{\partial}{\partial s} \gamma(s) \frac{\partial}{\partial s} + \gamma(s) k^2_{\beta}(s) \bigg]\ X(s, z) = N_b r_e \int_{\zeta}^{\infty} f_w(z') w_t(z' - z) X(s, z') dz' \\
\end{align*}
\]

Here \(\gamma(s)\) is the energy increase with distance due to acceleration, \(k_{\beta}(s)\) is the focusing, \(f_w\) is the longitudinal distribution in the bunch.

Assume \(\gamma(s) = \gamma_0 + gs\), \(k_{\beta}(s) = k_0 \sqrt{\gamma_0/\gamma(s)}\). If the focusing is due to plasma ions, then \(k_0 = k_p/\sqrt{2\gamma_0}\).

For simple models, there are analytical solutions to this equation.

Solution for a flat current profile

Assume a uniform current in the beam \( I \) and a strong focusing (the betatron wavelength is shorter than the gain length of the instability)

\[
\frac{X(s, z)}{X_0} \approx \frac{3^{1/4}}{2^{3/2} \pi^{1/2}} \left( \frac{\gamma_0}{\gamma(s)} \right)^{1/4} \frac{\exp(\Lambda)}{\Lambda^{1/2}} \cos \left( \theta_\beta - \frac{\Lambda}{3^{1/2}} + \frac{\pi}{12} \right)
\]

where

\[
\Lambda(s, z) = \frac{3^{3/2}}{2^{5/3}} \left( \frac{I}{I_A} \right)^{1/3} \left( \frac{w_t z^2}{g k_0} \right)^{1/6} \left( \frac{\gamma(s)}{\gamma_0} \right)^{1/6}
\]

An alternative approach is to solve the BBU equation numerically. This can be done for an arbitrary distribution function.
Numerical solution for a Gaussian bunch

Parameters of Weiming An simulations: the driver has $\sigma_z = 12.77$ µm, $\sigma_r = 3.65$ µm, $Q = 1.6$ nC, ($I_{\text{peak}} = 15$ kA); the witness has $\sigma_z = 6.38$ µm, $\sigma_r = 3.65$ µm, $Q = 0.69$ nC, ($I_{\text{peak}} = 13$ kA). Plasma density $4 \times 10^{16}$ cm$^{-3}$. The distance between the bunches is a) 108 µm and b) 150 µm.
One way to characterize BBU is to calculate the projected emittance:

\[
\epsilon_{\text{proj}}^2(s) = \langle (X - \bar{X})^2 \rangle \langle (X' - \bar{X}')^2 \rangle - \langle (X - \bar{X})(X' - \bar{X}') \rangle
\]

where the averaging means

\[
\langle \ldots \rangle = \int dz \ldots f_w(z)
\]
BBU instability—projected emittance

For a particular application this result can be translated into the jitter tolerance for the witness bunch.
Summary

- We used a combination of numerical and analytical tools to calculate the short-range wakefields inside a plasma bubble in the blowout regime of PWFA in axisymmetric geometry. For the wake calculation one needs to know the energy-density radial distribution in the bubble, which can be taken from 2D simulations of PWFA. These wakes are linear functions of the charge. They are stronger in the region of higher accelerating fields (closer to the end of the bubble). Simple fitting formulas for the wakes are derived.

- The wakes can be used in the standard formalism for the study of the beam breakup instabilities.

- We developed a matlab code that solves axisymmetric plasma bubble excited by a driver (runs a few minutes on a desktop computer).